PW2 Addendum: State Space Averaging for Advanced Digital Control of Power Converters

Kip Haggerty, Ph.D., P.E.

H&A Systems Engineering

Controls • Communications • Sensors

P.O. Box 2875 • El Segundo, CA 90245
(310) 607-9609 • http://www.hasys.com/

Download at: http://www.hasys.com/tutorials.html

© 2004 Kip Haggerty, ARR

Outline

- Design Approach
- State Space Averaging Methodology
- Buck Converter Example
- Advanced Algorithms

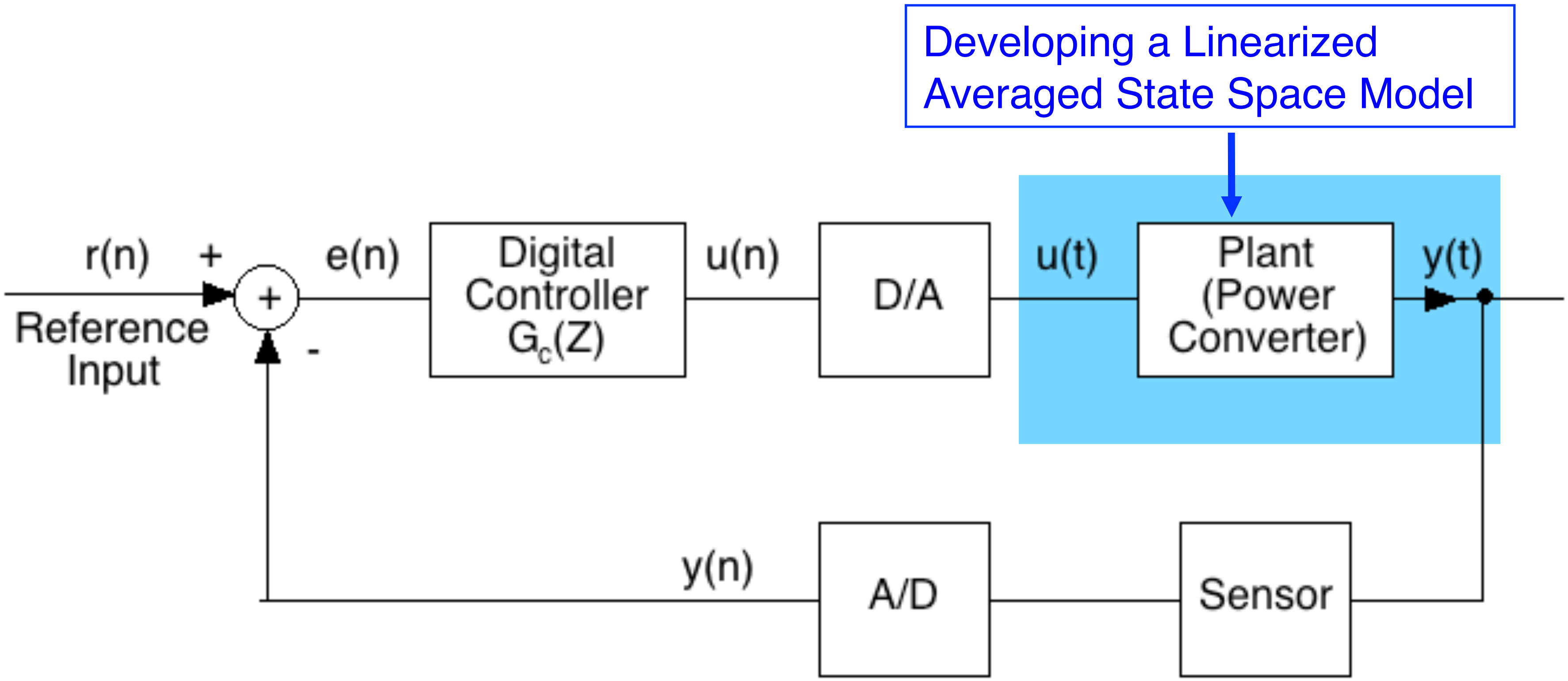
© 2004 Kip Haggerty, ARR

Recall Our Design Approach

- Analyze "Plant" (i.e. the Power Converter) in Analog (Continuous Time) Domain
  - Look for Type of Controller Needed
- Select Feedback Controller (PID, PI, Integrator, LPF, etc.)
- Analyze Continuous Time Closed Loop System with Pure Time Delay
  - Is Processing Interval Short Enough to Provide Desired Bandwidth?
- Convert Feedback Controller to Discrete Time
  - Check Digital Resolution
  - Check for Numerical Problems in Computations
  - Check if Resolution Expansion by Averaging is Required

© 2004 Kip Haggerty, ARR

Modeling Plant Required to Predict Performance



- **References:**
  - **Daniel M. Mitchell, <u>DC-DC Switching Regulator Analysis</u>, NY, McGraw-Hill, 1988. See Chapter 4.**
  - **[SB] Rudolf P. Severns and Gordon (Ed) Bloom, <u>Modern DC-to-DC Switchmode Power Converter Circuits</u>, NY, Van Nostrand, 1985. See Chapters 2 and 3.**

© 2004 Kip Haggerty, ARR

Plant Analysis Requires State Space Averaging and Small Signal Linearization

- **Develop First Order Matrix Differential Equations for Each Set of Switch Settings**
  - **Eq 1 On/Off for CCM and DCM**
  - **Eq 2 Off/On for CCM and DCM**
  - **Eq 3 Off/Off for DCM Only**
- **Take Average of Equations** → $\sum_i d_i \cdot [Eq]_i$
  - **d_i is the duty time in the ith case**
  - **Justified by High Switching Frequency**
- **Linearize Averaged Equation Via Small Signal Approximation**
  - **i.e. Linearize about DC Operating Point**
  - **Assume Any Product of Two ac Terms is Negligible**
  - **NOTE: Approximation May Be Invalid - Simulation/Prototyping**

- **References:**
  - **Daniel M. Mitchell, <u>DC-DC Switching Regulator Analysis</u>, NY, McGraw-Hill, 1988. See Chapter 4.**
  - **[SB] Rudolf P. Severns and Gordon (Ed) Bloom, <u>Modern DC-to-DC Switchmode Power Converter Circuits</u>, NY, Van Nostrand, 1985. See Chapters 2 and 3.**

Reminder: Circuit --> Linear Differential Equations --> Laplace Transform Algebraic Equations

Circuit Element	Differential Equation	Laplace Transform
Linear Constant Resistor Scale Factor	$v = R \cdot i$	$V(s) = R \cdot I(s)$
Linear Constant Inductor Differential Operator	$v = L \cdot \frac{di}{dt}$	$V(s) = s \cdot L \cdot I(s)$
Linear Constant Capacitor Integrator	$C \cdot \frac{dv}{dt} = i(t)$	$V(s) = \frac{1}{s \cdot C} \cdot I(s)$

- **We Ignore the Nonlinearity of the Switching by Treating the Switch States Separately**

Reminder: nth Order Linear Dif Eq Can Be Written As n-dimensional First Order Matrix Dif Eq

- **Standard Notation**

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\cdot\mathbf{x}(t) + \mathbf{B}\cdot\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t)$$

- **Transfer Function**

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\cdot\left[\mathbf{x}(0) + \mathbf{B}\cdot\mathbf{U}(s)\right]$$

$$\mathbf{Y}(s) = \mathbf{C}\cdot\mathbf{X}(s) + \mathbf{D}\cdot\mathbf{U}(s)$$

$$= \mathbf{C}\cdot(s\mathbf{I} - \mathbf{A})^{-1}\cdot\mathbf{x}(0) +$$

$$\left[\mathbf{C}\cdot(s\mathbf{I} - \mathbf{A})^{-1}\cdot\mathbf{B} + \mathbf{D}\right]\cdot\mathbf{U}(s)$$

$$\mathbf{G}(s) = \left[\mathbf{C}\cdot(s\mathbf{I} - \mathbf{A})^{-1}\cdot\mathbf{B} + \mathbf{D}\right]$$

- **DC Solution**

$$\frac{d}{dt}\mathbf{x} = 0$$

$$X = \mathbf{A}^{-1}\cdot\mathbf{B}\cdot U$$

$$Y = \left[\mathbf{C}\cdot\mathbf{A}^{-1}\cdot\mathbf{B} + \mathbf{D}\right]\cdot U$$

- **Buck Converter Definitions**

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix}$$

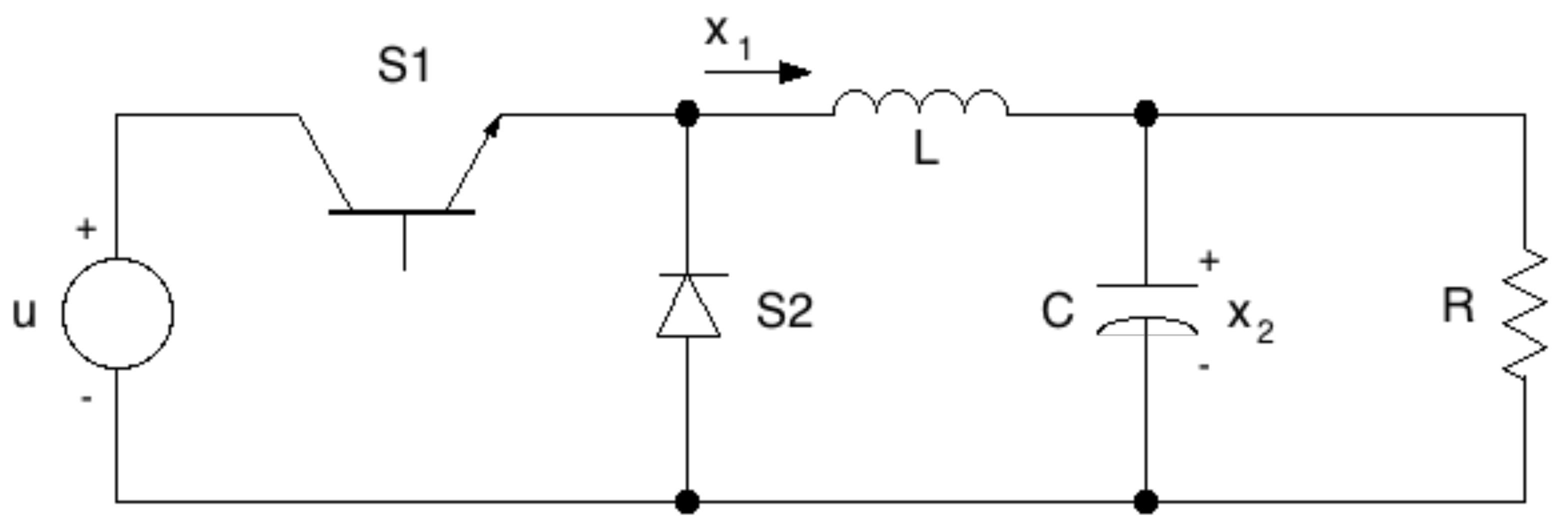
$$\mathbf{u}(t) = v_i(t)$$

$$\mathbf{y}(t) = v_o(t) = v_C(t)$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\mathbf{C}}\cdot\mathbf{x}(t) + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}}\cdot\mathbf{u}(t)$$

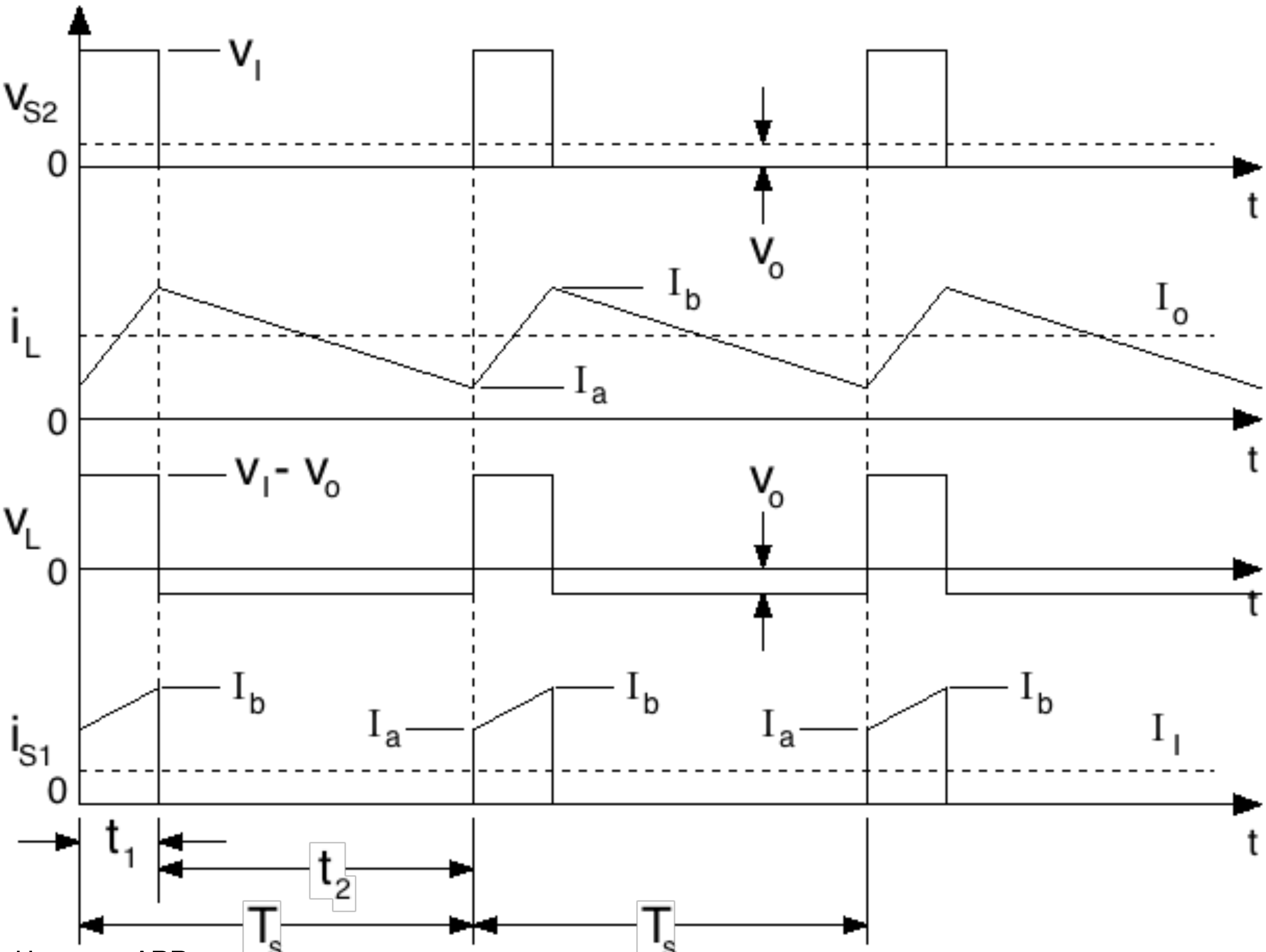
© 2004 Kip Haggerty, ARR

Buck Converter Circuit: Converts from High to Low Voltage DC



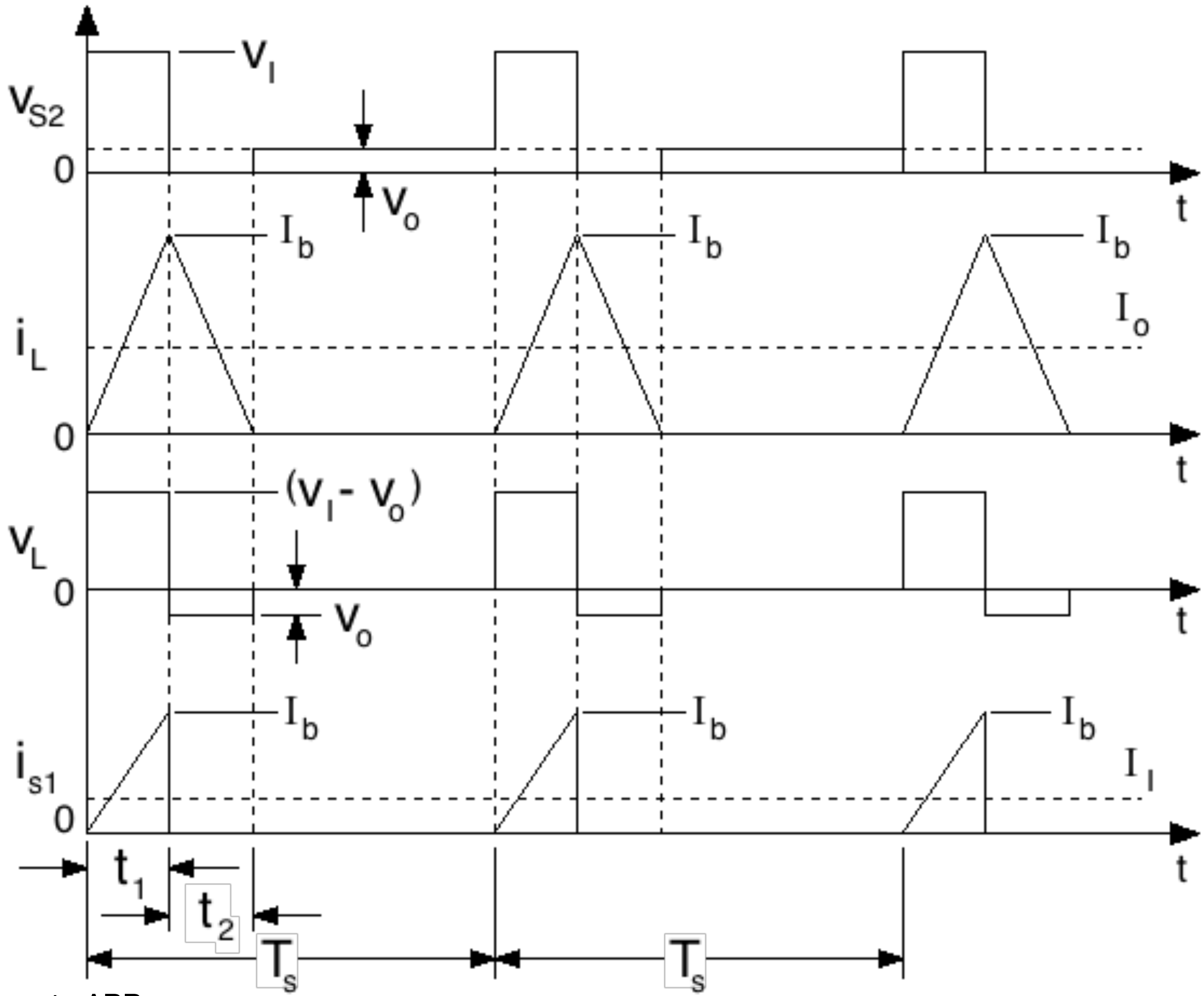
© 2004 Kip Haggerty, ARR

Continuous Current Mode (CCM) Waveforms



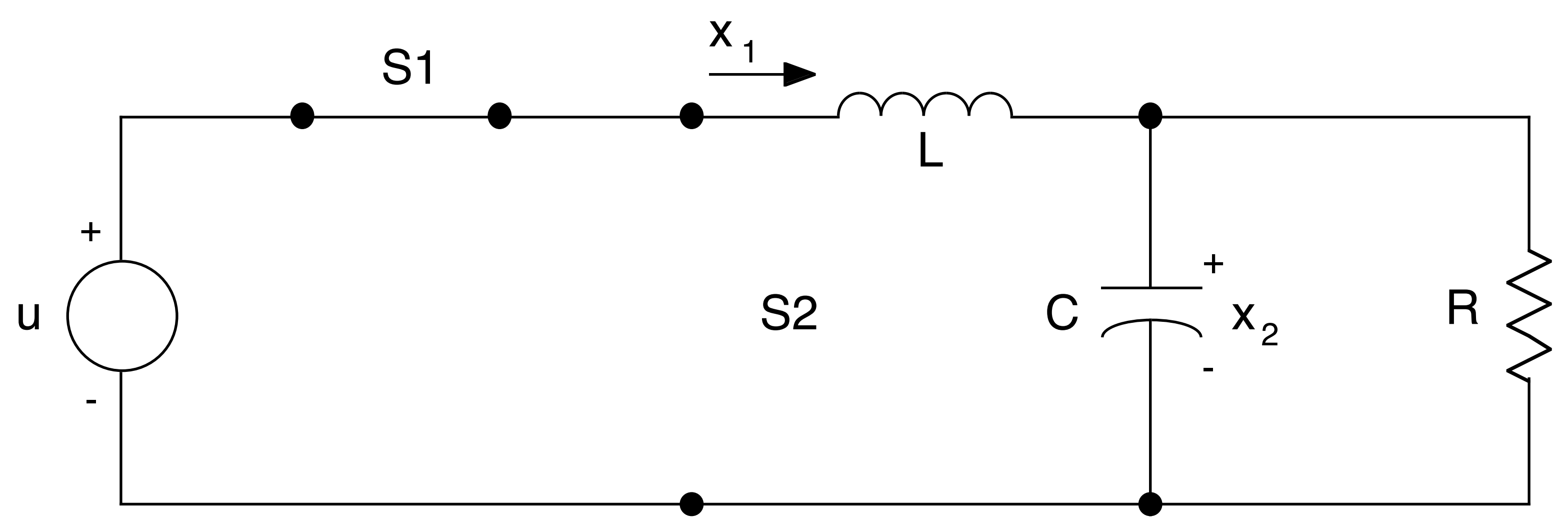
© 2004 Kip Haggerty, ARR

Discontinuous Current Mode (DCM) Waveforms



© 2004 Kip Haggerty, ARR

Case 1: Buck Converter Circuit
S1 On and S2 Off



© 2004 Kip Haggerty, ARR

Case 1: S1 On and S2 Off for CCM and DCM

- **Circuit Equations**

$$v_i(t) = L \cdot \tfrac{d}{dt} i_L(t) + v_C(t)$$

$$C \cdot \tfrac{d}{dt} v_C = i_L(t) - \frac{v_C(t)}{R}$$

- **In State Variables**

$$u(t) = L \cdot \tfrac{d}{dt} x_1 + x_2(t)$$

$$C \cdot \tfrac{d}{dt} x_2 = x_1(t) - \frac{1}{R} \cdot x_2(t)$$

- **Rearranging**

$$\tfrac{d}{dt} x_1 = -\frac{1}{L} \cdot x_2(t) + \frac{1}{L} \cdot u(t)$$

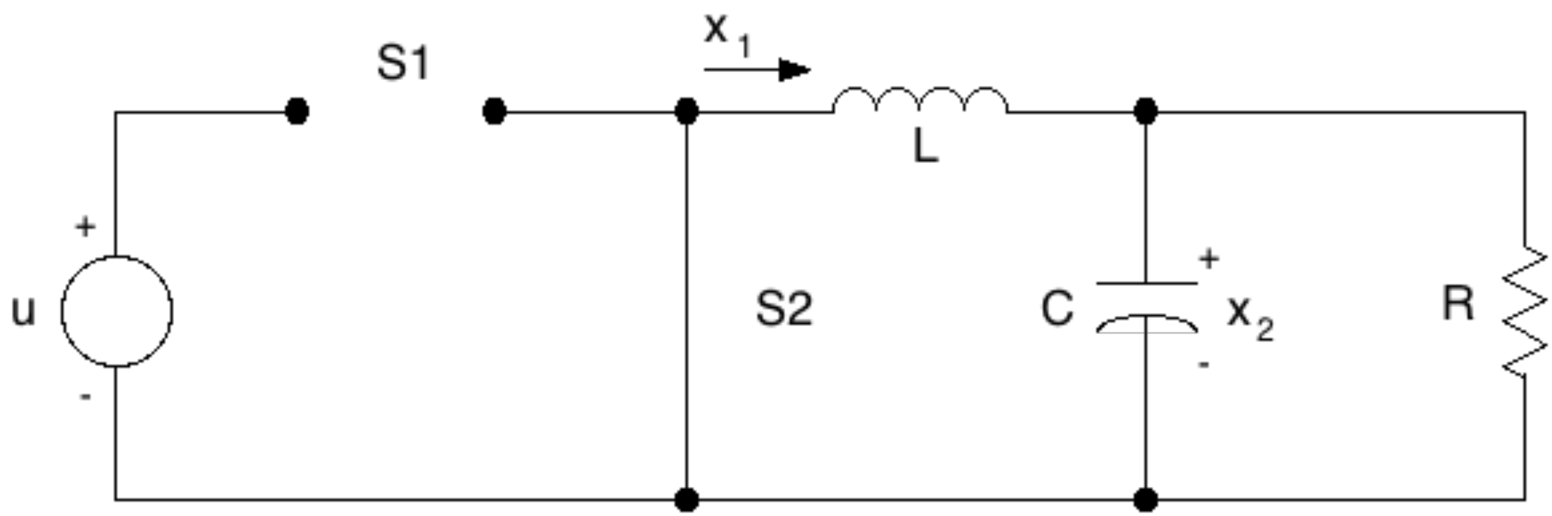
$$\tfrac{d}{dt} x_2 = \frac{1}{C} \cdot x_1(t) - \frac{1}{R \cdot C} \cdot x_2(t)$$

- **State Model**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R \cdot C} \end{bmatrix}}_{\mathbf{A_1}} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 1/L \\ 0 \end{bmatrix}}_{\mathbf{B_1}} \cdot \begin{bmatrix} u(t) \end{bmatrix}$$

© 2004 Kip Haggerty, ARR

Case 2: Buck Converter Circuit
S1 Off and S2 On



© 2004 Kip Haggerty, ARR

Case 2: S1 Off and S2 On for CCM and DCM

- **Circuit Equations**

$$0 = L \cdot \tfrac{d}{dt} i_L(t) + v_C(t)$$

$$C \cdot \tfrac{d}{dt} v_C = i_L(t) - \frac{v_C(t)}{R}$$

- **In State Variables**

$$0 = L \cdot \tfrac{d}{dt} x_1 + x_2(t)$$

$$C \cdot \tfrac{d}{dt} x_2 = x_1(t) - \frac{1}{R} \cdot x_2(t)$$

- **Rearranging**

$$\tfrac{d}{dt} x_1 = -\frac{1}{L} \cdot x_2(t)$$

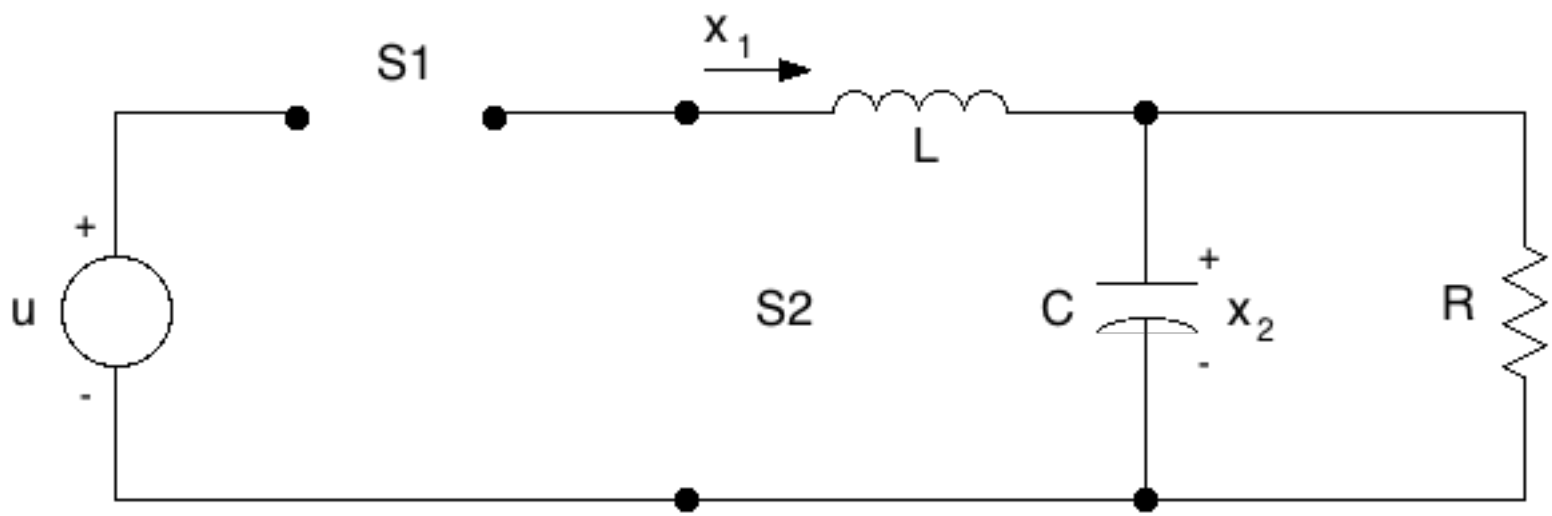
$$\tfrac{d}{dt} x_2 = \frac{1}{C} \cdot x_1(t) - \frac{1}{R \cdot C} \cdot x_2(t)$$

- **State Model**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R \cdot C} \end{bmatrix}}_{\mathbf{A_2}} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{B_2}} \cdot [u(t)]$$

© 2004 Kip Haggerty, ARR

Case 3: Buck Converter Circuit
S1 Off and S2 Off



© 2004 Kip Haggerty, ARR

Case 3: S1 Off and S2 Off for DCM Only

- **Circuit Equations**

$$i_L(t) = 0 \Rightarrow \tfrac{d}{dt} i_L(t) = 0$$

$$C \cdot \tfrac{d}{dt} v_C = 0 - \frac{v_C(t)}{R}$$

- **In State Variables**

$$\tfrac{d}{dt} x_1 = 0$$

$$C \cdot \tfrac{d}{dt} x_2 = 0 - \frac{1}{R} \cdot x_2(t)$$

- **Rearranging**

$$\tfrac{d}{dt} x_1 = 0$$

$$\tfrac{d}{dt} x_2 = 0 \cdot x_1(t) - \frac{1}{R \cdot C} \cdot x_2(t)$$

- **State Model**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{R \cdot C} \end{bmatrix}}_{A_3} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{B_3} \cdot [u(t)]$$

© 2004 Kip Haggerty, ARR

State Space Averaged System of Equations for CCM and DCM

- **CCM**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R \cdot C} \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d/L \\ 0 \end{bmatrix} \cdot [u(t)]$$

B(d) = d B_1+ (1 - d) B_2

A(d) = d A_1 + (1 - d) A_2

- **DCM**

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{d_1 + d_2}{L} \\ \frac{d_1 + d_2}{C} & -\frac{1}{R \cdot C} \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d_1/L \\ 0 \end{bmatrix} \cdot [u(t)]$$

A(d_1, d_2) = d_1 A_1 + d_2 A_2 + (1 - d_1 - d_2) A_3

B(d_1, d_2) = d_1 B_1 + d_2 B_2 + (1 - d_1 - d_2) B_3

© 2004 Kip Haggerty, ARR

Averaged CCM State Space Model: Linearization by Small Signal Analysis

- **Small Signal Equations**

DC AC

$$\mathbf{x} = X + \chi$$

$$\mathbf{u} = U + \mu$$

$$d = D + \delta$$

- **Substitute and Expand**

$$\tfrac{d}{dt}\mathbf{x} = \tfrac{d}{dt}X + \tfrac{d}{dt}\chi = \tfrac{d}{dt}\chi$$

$$= \mathbf{A}(D)\cdot X + \mathbf{B}(D)\cdot U$$

dc Solution

$$+\mathbf{A}(D)\cdot \chi + \mathbf{B}(D)\cdot \mu$$

$$= \mathbf{A}(\delta)\cdot X + \mathbf{B}(\delta)\cdot U$$

ac Approx

$$\cancel{+\mathbf{A}(\delta)\cdot \chi + \mathbf{B}(\delta)\cdot \mu}$$

Neglect ac Cross Terms

- **After Some Algebra**

$$\tfrac{d}{dt}\mathbf{x} = \mathbf{A}(D)\cdot \mathbf{x} + \mathbf{B}(D)\cdot \mathbf{u} + \mathbf{E}_{12}\cdot \delta$$

$$where: \quad \mathbf{E}_{12} = (\mathbf{A}_1 - \mathbf{A}_2)\cdot X + (\mathbf{B}_1 - \mathbf{B}_2)\cdot U$$

© 2004 Kip Haggerty, ARR

Averaged DCM State Space Model: Linearization by Small Signal Analysis

- **Small Signal Equations**

DC AC

$$\mathbf{x} = X + \chi$$

$$\mathbf{u} = U + \mu$$

$$d_1 = D_1 + \delta_1$$

$$d_2 = D_2 + \delta_2$$

- **Substitute and Expand**

$$\tfrac{d}{dt}\mathbf{x} = \tfrac{d}{dt}X + \tfrac{d}{dt}\chi = \tfrac{d}{dt}\chi$$

$$= \mathbf{A}(D_1, D_2)\cdot X + \mathbf{B}(D_1, D_2)\cdot U$$

dc Solution

$$+\mathbf{A}(D_1, D_2)\cdot \chi + \mathbf{B}(D_1, D_2)\cdot \mu$$

$$+\mathbf{A}(\delta_1, \delta_2)\cdot X + \mathbf{B}(\delta_1, \delta_2)\cdot U$$

ac Approx

$$\cancel{+\mathbf{A}(\delta_1, \delta_2)\cdot \chi + \mathbf{B}(\delta_1, \delta_2)\cdot \mu}$$

Neglect ac Cross Terms

- **After Some Algebra**

$$\tfrac{d}{dt}\mathbf{x} = \mathbf{A}(D_1, D_2)\cdot \mathbf{x} + \mathbf{B}(D_1, D_2)\cdot \mathbf{u}$$

$$+ \mathbf{E}_{13}\cdot\delta_1 + \mathbf{E}_{23}\cdot\delta_2$$

$$where:$$

$$\mathbf{E}_{13} = (\mathbf{A}_1 - \mathbf{A}_3)\cdot X + (\mathbf{B}_1 - \mathbf{B}_3)\cdot U$$

$$\mathbf{E}_{23} = (\mathbf{A}_2 - \mathbf{A}_3)\cdot X + (\mathbf{B}_2 - \mathbf{B}_3)\cdot U$$

© 2004 Kip Haggerty, ARR

Now Let's Get Specific

- **Component Values**

$$U = V_I = 10\ Volts$$

$$V_O = 5\ Volts$$

$$L = 100\ \mu H$$

$$C = 1{,}000\ \mu F$$

$$T_S = 10\ \mu s$$

$$R = \left\langle \begin{matrix} 1\,\Omega & CCM \\ 100\,\Omega & DCM \end{matrix} \right\rangle$$

- **Thus**

$$M = \frac{V_O}{V_I} = 0.5$$

$$\tau_{LC} = \frac{1 - M}{2} = 0.25$$

$$\tau_L = \frac{L}{R \cdot T_S} = \left\langle \begin{matrix} 10 > \tau_{LC} & CCM \\ 0.1 < \tau_{LC} & DCM \end{matrix} \right\rangle$$

© 2004 Kip Haggerty, ARR

The DC Solution is:

- **CCM**

$$D = M = 0.5$$

$$U = V_I = 10$$

$$X = \begin{bmatrix} I_L \\ V_C \end{bmatrix} = \begin{bmatrix} D/R \\ D \end{bmatrix} \cdot V_I = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

- **DCM**

$$D_1 = M\sqrt{\frac{2 \cdot \tau_L}{1 - M}} = 0.316$$

$$D_2 = \sqrt{2 \cdot \tau_L \cdot (1 - M)} = 0.316$$

$$U = V_I = 10$$

$$X = \begin{bmatrix} I_L \\ V_C \end{bmatrix} = \begin{bmatrix} \dfrac{D_1}{(D_1 + D_2)^2 \cdot R} \\ \dfrac{D_1}{D_1 + D_2} \end{bmatrix} \cdot V_I = \begin{bmatrix} 7.91 \\ 5 \end{bmatrix}$$

© 2004 Kip Haggerty, ARR

The Open Loop Transfer Function Is:

- **The CCM Transfer Functions Are:**

$$G_b(s) = 0.5\,\frac{1 - 2\cdot 10^{-3}\cdot s}{1 - 1.6\cdot 10^{-3}\cdot s}$$

$$G_e(s) = \frac{1 - 2\cdot 10^{-3}\cdot s}{1 - 1.6\cdot 10^{-3}\cdot s}$$

- **Feedback is Unity**

$$H(s) = 1$$

- **Throw in Pure Time Delay**

$$G_d(s) = e^{-s\cdot T_d}$$

- **Use First Order Controller Simple Integrator**

$$G_c(s) = k\frac{1}{s}$$

- **Open Loop Transfer Function Is:**

$$GH(s) = G_c(s)\cdot G_e(s)\cdot G_d(s)\cdot H(s)$$

$$= k\frac{1}{s}\cdot\frac{1 - 2\cdot 10^{-3}\cdot s}{1 - 1.6\cdot 10^{-3}\cdot s}\cdot e^{-s\cdot T_d}$$

- **Now We Can Do Some Bode Plots**

Linearized Averaged State Space Model Enables Use of Modern Algorithms

- One Such Algorithm is the Kalman Filter
  - Powerful Estimation Method
  - Linear unbiased minimum variance estimate
  - Maximum likelihood estimate (Linear and Gaussian)
  - Works on steadily expanding set of data
  - Recursive
- Where Would One Be Used
  - Estimate States for Feedback Control
  - For Buck Converter
    - » Inductor Current (Running Average)
- The Next Few Charts Provide a Brief Intro to the Equations for the Discrete Time Kalman Filter

© 2004 Kip Haggerty, ARR

Discrete Time Model of Plant

$$x_{n+1} = \Phi(n+1,n)\,x_n + B_n u_n + G N_n^s$$

$$z_n = H_n x_n + R^{1/2} N_n^z$$

$$E\left[N_n^s\right] = 0$$

$$E\left[N_n^s N_m^{sT}\right] = I\delta_{nm}$$

$$E\left[N_n^z\right] = 0$$

$$E\left[N_n^z N_m^{zT}\right] = I\delta_{nm}$$

© 2004 Kip Haggerty, ARR

Kalman Filter Equations (Recursive)

Gain: $K_n = P_{p_n} H_n' \left[H_n P_{p_n} H_n' + R_n \right]^{-1}$

Filter: $x_{f_n} = x_{p_n} + K_n \left(z_n - H_n x_{p_n} \right)$

Prediction: $x_{p_{n+1}} = \Phi(n+1, n)\, x_{f_n} + B_n u_n$

Filter Cov.: $P_{f_n} = \left[I - K_n H_n \right] P_{p_n}$

Prediction: $P_{p_n} = \Phi(n+1, n)\, P_{f_n} \Phi(n+1, n)^T + Q$

where:

$$x_{f_n} = \hat{x}_{n|n} = E\left[x_n \,|\, z_n \ldots, z_1 \right]$$

$$x_{p_n} = \hat{x}_{n|n-1} = E\left[x_n \,|\, z_{n-1} \ldots, z_1 \right]$$

$$P_{f_n} = \mathrm{Cov}\left(x_n - \hat{x}_{n|n} \right) = E\left[\left(x_n - \hat{x}_{n|n} \right) \left(x_n - \hat{x}_{n|n} \right)^T \right]$$

$$P_{p_n} = \mathrm{Cov}\left(x_n - \hat{x}_{n|n-1} \right) = E\left[\left(x_n - \hat{x}_{n|n-1} \right) \left(x_n - \hat{x}_{n|n-1} \right)^T \right]$$

© 2004 Kip Haggerty, ARR

Initialization Case 1: Know Initial Mean and Cov

$$\bar{x} = E[x_0]$$

$$P_0 = Cov(x_0 - \bar{x}_0) = E\left[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\right]$$

Set $x_p = \bar{x}_0$

$P_p = P_0$

Then run filter

Initialization Case 2: No A Priori Information

$$\text{Set}: x_f = f1(z_0, \ldots)$$

$$P_f = f2(R)$$

Then compute :

$$x_p = \Phi x_f + Bu_0$$

$$P_p = \Phi P_f \Phi^T + Q$$

Then run filter

$f1(z_0, \ldots)$ maps measurement(s) to state

$f2(R)$ maps measurement cov to filter cov

Note: May want to guess some values rather than collecting several estimates to "measure" full set of states.

© 2004 Kip Haggerty, ARR